A CLASS OF IRRATIONALS

SECOLINSKY

Our first fact to establish will be used to prove that a certain class of numbers is irrational.

Lemma 1. $(\forall n \in \mathbb{Z}_+)(n \geq 2 \Rightarrow 2n > n+1)$

Proof. We will use the Principle of Induction. Let $p(n) := 2n > n+1$. We first have that $2(2) = 4 > 3 = 2+1$. So $p(2)$ is true. Now assuming $p(m)$ to be true, we'll want to show that $p(m+1)$ is true. From our assumption, it follows then that $2m > m+1$ implies that $2m+2 > (m+1)+2$ and thus we have that $2(m+1) = 2m+2 > (m+1)+2 > (m+1)+1$. Therefore we have shown that $p(m)$ implies $p(m+1)$. □

Now to use our result.

Theorem 1. $(\forall n \in \mathbb{Z}_+)(n \geq 2 \Rightarrow \sqrt[n]{n}$ *is irrational*$)$

Proof. We'll first show using the Principle of Induction that for all n greater than or equal to two, $n < 2^n$. Let $\mathcal{M} = \{n \in \mathbb{Z}_+ : n < 2^n\}$. We first observe that $2 < 4 = 2^2$, thus $2 \in \mathcal{M}$. Now let $p \in \mathcal{M}$ so that we have that $p < 2^p$ implies that $2p < 2(2^p) = 2^{p+1}$. Using Lemma 1, it then follows that $p+1 < 2p$ and so $p+1 < 2^{p+1}$. Thus if $p \in \mathcal{M}$, then $p+1 \in \mathcal{M}$. It is therefore established that for all $n \in \mathbb{Z}_+$ greater than or equal to two, $n < 2^n$. It then follows that $1^n < n < 2^n$ implies $1 < \sqrt[n]{n} < 2$.

Now assume to the contrary that $\sqrt[n]{n}$ is not irrational, that it can be expressed as a rational $\frac{m}{k}$ in reduced form where $k > 1$. From the Fundamental Theorem of Arithmetic, the prime factorization of m and k have no common prime factors because if they did, then $\frac{m}{k}$ would not be in reduced form as is assumed. So we have that for the n^{th} power of m and k, $n = \left(\frac{m}{k}\right)^n \notin \mathbb{Z}$. This is a contradiction. Hence, $\sqrt[n]{n}$ must be irrational. □