THE CAUCHY-SCHWARZ INEQUALITY

SECOLINSKY

A few definitions, lemmas, and the Cauchy-Schwarz inequality are introduced to show that the Euclidean norm is a metric.

Definition 1. *Let $x \in \mathbb{R}^n$. The Euclidean norm of x is $\sqrt{\sum_1^n x_i^2}$ and denoted as $\|x\|$.*

Definition 2. *A metric on a set X is a real-valued function $d : X \times X \to \mathbb{R}$ with the following properties*:

(i) $d(x, y) \geq 0$,
(ii) $d(x, y) = 0$ *if and only if* $x = y$,
(iii) $d(x, y) = d(y, x)$, *and*
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Properties (i)-(iii) of a metric are easy to verify for the Euclidean norm, but property (iv), the triangle inequality, requires mental push ups which we now begin.

Lemma 1. *Let $a, b, c \in \mathbb{R}$ such that $a\lambda^2 + b\lambda + c \geq 0$ for all $\lambda \in \mathbb{R}$. Then $b^2 - 4ac \leq 0$.*

Proof. Because our polynomial is greater or equal to zero, the root(s) $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ can be either on the λ-axes or imaginary for which cases we have $b^2 - 4ac = 0$ and $b^2 - 4ac < 0$ respectively. □

Theorem 1 (Cauchy-Schwarz Inequality)**.** *Let $x, y \in \mathbb{R}^n$. $|\sum_1^n x_i y_i| \leq (\sum_1^n x_i^2)^{\frac{1}{2}} (\sum_1^n y_i^2)^{\frac{1}{2}}$.*

Proof. Notice that $0 \leq \sum_1^n (x_i - \lambda y_i)^2 = \lambda^2 \sum_1^n y_i^2 - 2\lambda \sum_1^n x_i y_i + \sum_1^n x_i^2$ for all $\lambda \in \mathbb{R}$. From the lemma above it follows that $(2\sum_1^n x_i y_i)^2 - 4(\sum_1^n y_i^2)(\sum_1^n x_i^2) \leq 0$. So $4(\sum_1^n x_i y_i)^2 \leq 4(\sum_1^n y_i^2)(\sum_1^n x_i^2)$ implies $(\sum_1^n x_i y_i)^2 \leq \|y\|^2 \|x\|^2$ and thus $|\sum_1^n x_i y_i| \leq \|y\| \|x\|$. □

Lemma 2. $(\forall x, y \in \mathbb{R}^n)(\ \|x + y\| \leq \|x\| + \|y\|\ )$

Proof. Consider $x, y \in \mathbb{R}^n$ and note that $\|x + y\|^2 = \sum_1^n (x_i + y_i)^2 = \sum_1^n x_i^2 + 2\sum_1^n (x_i y_i) + \sum_1^n y_i^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$. Taking the square root of both sides yields the desired result of $\|x + y\| \leq \|x\| + \|y\|$. □

The Main Statement

Theorem 2. *Let $x, y \in \mathbb{R}^n$ where $d(x, y) := \sqrt{\sum_1^n (x_i - y_i)^2}$. Then d is a metric on $\mathbb{R}^n$.*

Proof. Since $(x_i - y_i)^2$ is always computed for every i^{th} coordinate of x and y and then summed, $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$. It's also clear that $d(x, y) = 0$ if and only if $x = y$. To show the last property of a metric, let $z \in \mathbb{R}^n$ and observe that $\|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|$. Hence, $d(x, z) \leq d(x, y) + d(y, z)$. □