POKER HAND ENUMERATIONS

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The concept of multiset permutations will be used to help count all possible poker hands. It is a concept which abstracts the notion of a set. It's important we be familiar and comfortable with such an object. So we begin. We say that a multiset is a collection of objects from a nonempty set S with repetitions allowed and with disregard to order. The concept is more formally defined.

Definition 1. *Let $\mathbb{Z}_+$ be the set of nonnegative integers and for a nonempty set S, consider $m : S \to \overline{\mathbb{Z}}_+$. A multiset is the pair (S, m) and written as $\{m(s)s : s \in S\} = M$. We call m the multiplicity function and call a linear order which contains all the elements of M a multiset permutation.*

From the definition we have that the size of M is equal to $\sum_{s\in S} m(s)$. Say the size of M is n and that there are $\mathcal{A}$ multiset permutations. Obviously, if we decide to treat M as a set of n distinct objects, we'd have $n!$ permutations. The issue is how to obtain the number $n!$ from only knowing $\mathcal{A}$. The answer comes from thinking about how an individual multiset permutation a yields $a \prod_{s\in S} m(s)!$ permutations. We then have $\mathcal{A} \prod_{s\in S} m(s)! = n!$ which gives the following result that the number $\mathcal{A}$ of multiset permutations of M is

$$\frac{n!}{\prod_{s\in S} m(s)!}$$

What now follows are two concrete examples which force us to differentiate between multisets and sets.

Example 1. *Consider the multiset $A = \{1, 2, 2, 3, 4, 5, 6\}$ and the number $\mathcal{A}$ of multiset permutations of A that don't have consecutive identical digits. Observe that $\mathcal{A}$ is the subtraction of the number of multiset permutations with the two objects 2 consecutively placed from the total number of all possible multiset permutations. Hence, $\mathcal{A}$ is equal to $1,800$*

$$= \frac{7!}{1!2!1!1!1!1!} - \frac{6!}{1!1!1!1!1!1!}.$$

Example 2. *An employee is going to work five days of the week with at least one of the two last days of the week off where the days of the week are represented as the set S_8, the set of positive integers less than eight. Given that $\Omega = \{A \subset S_8 : |A| = 5\}$ and B is the set of all 7-digit binary strings with 5 ones and 2 zeros, we define a function $f : \Omega \to B$ that maps A to x such that x_i is 1 if $i \in A$ and 0 otherwise. Clearly f is a bijection and so counting the right elements of B will give us the permissable five-day work week configurations for the employee. Interpreting 1 as a day of work and 0 as a day of rest, we have the following situations:*

1. *In the case of $x_6 = x_7 = 0$, there are $\binom{5}{5}$ ways to choose where to place the 5 ones*
2. *In the case of $x_6 = 1$ and $x_7 = 0$, there are $\binom{5}{4}$ ways to choose the remaining 4 ones.*
3. *In the case of $x_6 = 0$ and $x_7 = 1$, there are $\binom{5}{4}$ ways to choose the remaining 4 ones.*

Therefore, there are $\binom{5}{5} + 2\binom{5}{4} = 1 + 2(5) = 11$ permissable five-day work week configurations for our employee.

We have now developed our theoretical toolbox to be ready to deal with enumerating all poker hands. The game of poker has 13 denominations and four suits for each denomination and thus, from the multiplication principle, there are $13(4) = 52$ cards in a poker deck. Now consider the 52 choose 5 hands of a poker game.

There are 4 royal flush hands since for each of the four suites, there is one royal flush, namely a hand with denominations $10, J, Q, K, A$.

A straight hand requires having five denominations so as to be consecutive when ordered. There are two hands with nonnumeric denominations with the prescribed order of $10, J, Q, K, A$ and $A, 2, 3, 4, 5$. Taking five of the fourteen positions of the multiset permutation $(A, 2, \ldots, 10, J, Q, K, A)$ and ordering them as $a_1, \ldots, a_5$ will give us the positions of the five denominations required for a straight if they satisfy $1 \leq a_1 = a_2 - 1 = \ldots = a_5 - 4 \leq 10$. Thus there are ten possible configurations for choosing the five denominations at positions a_i of our multiset permutation that are permissable for a straight.

A straight flush has the property of being a straight with all denominations being of the same suite, but is not one of the four royal flush hands. Using the multiplication principle for counting, there are 36 such hands. Indeed, we have ten ways of choosing five permissable denominations, then four suites to make our flush. Subtracting the 4 royal flush hands from $10(4)$ yields our result.

To obtain a four-of-a-kind hand, first choose one of thirteen denominations from which we'll have all four suites. Then choose one of the remaining twelve denominations from which we'll choose one suite. Again applying the multiplication principle, it then follows that there are $624 = 13(12)4$ hands of such type.

A full house is obtained from first choosing one of the thirteen denominations and then choosing three of its four suites. Then one more denomination is picked for which we choose two of its four suites. We have then that the number of such hands is

$$13\binom{4}{3}12\binom{4}{2} = 13(4)12(6) = 3,744.$$

A flush is a hand with all five cards having the same suit, but not a royal flush nor a straight flush. There are $5,148$ hands that have their five cards all with the same suite since there are four ways to choose a suite, then 13 choose 5 ways to choose the denominations for that suite. Subtracting the 40 restricted hands from $5,148$ gives the number of flush hands to be $5,108$.

The property of a straight being ordered has been defined. Furthermore, to qualify as being a straight hand, it must not be one of the 36 straight flush hands or one of the 4 royal flush hands. The total number of hands with the property of being straight is $10,240 = 10(4^5)$ since there are 4 suites to choose from for each of the five possible denominations all belonging to one of the ten. Subtracting the 40 from $10,240$ gives us the number of straight hands to be $10,200$.

To obtain the number of three-of-a-kind, two pair, and one pair hands, let the multiplication principle for counting speak for itself:

$$54,912 = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2,$$
$$123,552 = \binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1},$$
$$1,098,240 = \binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3.$$

Adding everything mentioned so far, we come across there being $1,296,420$ hands. It was mentioned that there are $\binom{52}{5} = 2,598,960$ possible hands. Therefore, there are $(2,598,960 - 1,296,420) = 1,302,540$ hands which are given no special name and are ones we hope not to have.