TRIANGLE PARTITIONING AND INTEGER SAMPLING

SECOLINSKY

Let T be a triangle and $n \in \mathbb{N}$. Then T can be partitioned into $3n+1$ similar triangles.

Proof. Consider the midpoints of all sides of T. We then have 4 similar triangles $T_1, \ldots, T_4$ that partition T as shown for a regular triangle in Figure 1. Thus the statement is true for $n = 1$.

Now assume the statement is true for $n-1$ so that there are $3(n-1)+1$ similar triangles which partition T. Take any of the $3(n-1)+1$ triangles, call it x. Since x can be partitioned into 4 similar triangles, we have $3(n-1)+4 = 3n+1$ similar triangles which partition T. □

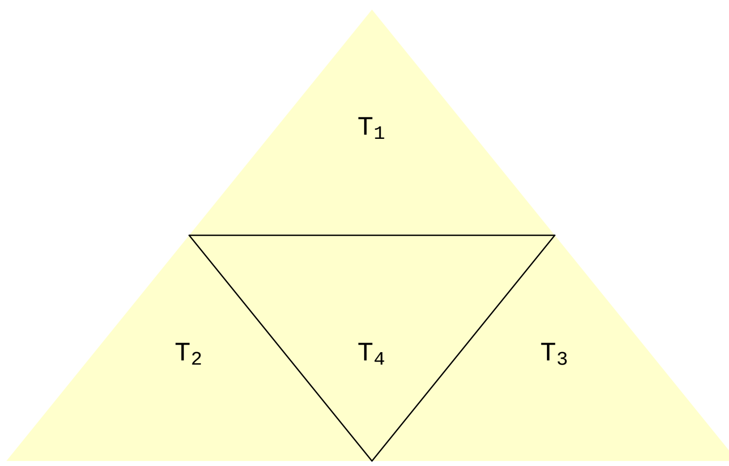

FIGURE 1

Given 17 points randomly thrown into an equilateral triangle of length 1, there exists two points such that the distance between them is at most $\frac{1}{4}$.

Proof. From the previous statement, there are $3(5)+1 = 16$ similar triangles $T_1, \ldots, T_{16}$ that partition our triangle T. Furthermore, each side of T is divided into 4 pieces. Since T is equilateral with sides of length 1 and all triangles are similar, the length of all sides of all triangles T_i is $\frac{1}{4}$. Therefore, there is one of the 16 triangles T_i with at least 2 points whose distance apart is at most $\frac{1}{4}$. □

Let $n \in \mathbb{N}$ and $N = \{1, \ldots, 2n\}$. If a random sample of $n+1$ distinct elements from N are taken, then there are elements x and y in the sample for which x is a multiple of y.

Proof. To show the validity of statement 1.22, it'll first be shown that for all integers m, there exists a nonnegative integer k and an odd number a such that $m = 2^k a$.

If m is odd, then let $k = 0$ and $a = m$. Otherwise, let m be even so that $m = 2^k q$ for some integer q which has no factors of 2. From the Fundamental Theorem of Arithmetic there is a unique prime factorization of q and because q has no factors of 2, it must be odd.

It is correct to state now that for all $m \in N$, there's an odd a and a nonnegative integer k such that $m = 2^k a$. To obtain all elements of N, it must be that $1 \leq a \leq 2n - 1$, i.e. there are only n choices for a.

From the Pigeon-Hole Principle, there are two elements in the sample of $n + 1$ which have the same a, call them x and y. Let $x = 2^t a$, $y = 2^k a$ and, without loss of generality, $t > k$. It then follows that x is a multiple of y. □

The statement that there are two of the $n + 1$ sampled elements such that one is double the other is false. Let S_n be the section of positive integers less than n. If we sample a random subset of size six from the universe S_{11}, it's possible to come across the following counter example: $\{2, 7, 6, 8, 9, 10\}$.